Dissipative stability theory for linear repetitive processes with application in iterative learning control

Wojciech Paszke, Paolo Rapisarda, Eric Rogers, Maarten Steinbuch

***Abstract*— This paper develops a new set of necessary and sufficient conditions for the stability of linear repetitive processes, based on a dissipative setting for analysis. These conditions reduce the problem of determining whether a linear repetitive process is stable or not to that of checking for the existence of a solution to a set of linear matrix inequalities (LMIs). Testing the resulting conditions only requires computations with matrices whose entries are constant in comparison to alternatives where frequency response computations are required.**

I. Introduction

Linear repetitive processes are one of the most important classes of two-dimensional (2D) linear systems and are of both practical and algorithmic interest. The unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t), 0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t),\ 0 \leq t \leq \alpha,\ k \geq 0$.

Physical examples of these processes include long-wall coal cutting and metal rolling operations [7], [6]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of such algorithmic applications include classes of iterative learning control schemes [4] and iterative algorithms for solving nonlinear dynamic optimal stabilization problems based on the maximum principle [5]. In this latter case, for example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient iterative solution algorithms and in the former it provides a stability theory which, unlike many alternatives, provides information concerning an absolutely critical problem in this application area, i.e. the trade-off between convergence and the learnt dynamics.

Attempts to control these processes using standard (or 1D) systems theory and algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass (k direction) and along a given pass (t direction) and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [6] based on an abstract model of the dynamics in a Banach space setting which includes a very large class of processes with linear dynamics and a constant pass length as special cases. Also the results of applying this theory to a range of sub-classes, including those considered here, have been reported [6].

The case of 2D discrete linear systems recursive in the positive quadrant $(i, j),\ i \geq 0,\ j \geq 0$, (where i and j denote the directions of information propagation) has been the subject of much research effort over the years using, in the main, the well known Roesser and Fornasini Marchesini state-space models (for the original references see [6]). It is natural, therefore, to ask if linear repetitive processes can be analyzed using this theory. In the case of discrete linear repetitive processes where the dynamics in both directions are governed by difference equations, it can be shown that an equivalence exists in terms of stability, but this is critically dependent on the structure of the boundary conditions. Moreover, there are other systems theoretic questions for discrete linear repetitive processes which have no counterparts in 2D discrete linear systems. A detailed treatment of this general area can be found in [6]. Also in differential linear repetitive processes information propagation along the pass is governed by a matrix differential equation and 2D discrete linear systems theory is not applicable.

Recognizing the unique control problem, the stability theory [6] for linear repetitive processes is of the bounded-input bounded-output (BIBO) form, i.e. bounded inputs are required to produce bounded sequences of pass profiles (where boundedness is defined in terms of the norm on the underlying Banach space). Moreover, it consists of two concepts, one of which is defined over the finite pass length and the other is independent of this parameter. In particular, asymptotic stability guarantees this BIBO property over the finite and fixed pass length whereas stability along the pass is stronger since it requires this property uniformly, and hence it is not surprising that asymptotic stability is a necessary

W. Paszke and M. Steinbuch are with Control Systems Technology Group, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. {W.Paszke,M.Steinbuch}@tue.nl

P. Rapisarda and E. Rogers are with School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK, {rp3,etar}@ecs.soton.ac.uk

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condition for stability along the pass.

If asymptotic stability holds for a discrete or differential linear repetitive process then any sequence of pass profiles generated converges in the pass-to-pass direction to a limit profile which is described by a 1D differential or discrete linear systems state-space model respectively. The finite pass length, however, means that the resulting 1D linear system could have an unstable state matrix stable since over a finite duration even an unstable 1D linear system can only produce a bounded output. There are also applications such as that in [5] where asymptotic stability is all that can be achieved.

In cases where asymptotic stability is not acceptable, stability along the pass is required and for the processes considered here the resulting conditions can be tested by 1D linear systems tests. Such tests, however, do not lead on to effective control law design algorithms. For example, in the differential case it is required to test that all eigenvalues of an $m \times m$ transfer-function matrix $G(s)$, where m is the dimension of the pass profile vector, lie in the open unit circle in the complex plane $s = j\omega,\ \omega \geq 0$. This could clearly lead to a significant computational load and also, despite the Nyquist basis, does not provide a basis for control law design. The most effective control law design method currently available for both differential and discrete processes starts from a Lyapunov function interpretation and leads to LMI based stability tests and control law design algorithms but is based on sufficient but not necessary stability conditions.

In this paper we first develop new necessary and sufficient conditions for stability along the pass of differential and differential repetitive processes that can also be computed using LMIs. The results are based on dissipative theory and make extensive use of the Kalman-Yakubovich-Popov (KYP) lemma that allows us to establish the equivalence between the frequency domain inequality (FDI) for a transfer-function and an LMI defined in terms of its state-space realization [1], [3]. To employ the KYP lemma, we need the stability conditions expressed in the form of an FDI as a first step and this means that we must restrict attention to the single-input single-output (SISO) case. In which context, note that a large number of the practical examples of repetitive processes are SISO [6] and in the final part of this paper we give the first of application of them to repetitive process based analysis of ILC laws.

Throughout this paper, $\sigma(A)$ and $\rho(A)$ denote the spectrum and the spectral radius of a given matrix A. The null and identity matrices with appropriate dimensions are denoted by 0 and I, respectively. Furthermore, $M \succ 0$ (respectively, $\succeq 0$) denotes a real symmetric positive definite (respectively, semi-definite) matrix. Similarly, $M \prec 0$ (respectively, $\preceq 0$) denotes a real symmetric negative definite (respectively, semi-definite) matrix. Finally, the symbol $\mathbb{C}$ denotes the set of a complex numbers and $\mathbb{C}_-$ the open left-half of the complex plane.

II. Discrete and Differential Linear Repetitive Processes

A. Stability Theory

Following [7], the state-space model of a discrete linear repetitive process has the following form over $0 \leq p \leq \alpha - 1,\ k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + B_0 y_k(p) + Bu_{k+1}(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + D_0 y_k(p) + Du_{k+1}(p) \end{aligned} \tag{1}$$

where $\alpha < +\infty$ denotes the pass length, and on pass $k \geq 0$ $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}$ is the pass profile (output) and $u_k(p) \in \mathbb{R}^r$ is the input vector.

To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). For the purposes of this paper, it is assumed that the state initial vector at the start of each new pass is of the form $x_{k+1}(0) = d_{k+1},\ k \geq 0$, where the $n \times 1$ vector d_{k+1} has known constant entries. Also it is assumed that the initial pass profile $y_0(p)$ is equal to a known vector $f(p)$.

Again following [7], [6], the state-space model of a differential linear repetitive process has the following form over $0 \leq t \leq \alpha,\ k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + Bu_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t) + Du_{k+1}(t) \end{aligned} \tag{2}$$

where the dimensions of the vectors involved are as in the discrete case above. Finally, the boundary conditions are taken as $x_{k+1}(0) = d_{k+1},\ k \geq 0$, where the $n \times 1$ vector d_{k+1} has known constant entries, and $y_0(p) = f(t)$, where $f(t)$ is an $m \times 1$ vector with known entries.

In terms of the analysis in this paper, no loss of generality arises from assuming that the initial pass profile is the zero vector and also in both cases $d_{k+1} = 0,\ k \geq 0$. Hence we will make no further explicit reference to the boundary conditions in this paper.

Several sets of necessary and sufficient conditions for stability along the pass of both discrete and differential linear repetitive processes of the form considered here are known [6], and here we will make use of those given in terms of the corresponding 2D characteristic polynomial where for the discrete case this is defined as

$$\mathcal{C}_{\text{disLRP}}(z_1, z_2) = \det\left(\begin{bmatrix} I - z_1 A & -z_1 B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}\right) \tag{3}$$

where $z_1, z_2 \in \mathbb{C}$ are the inverses of z-transform variables in the along the pass and pass-to-pass directions respectively, again see [6] for the details concerning these transform variables and, in particular, how to avoid technicalities associated with the finite pass length, which define a 2D transfer-function matrix for these processes.

For the differential processes, the 2D characteristic polynomial is

$$\mathcal{C}_{\text{difLRP}}(s, z_2) = \det\left(\begin{bmatrix} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}\right) \tag{4}$$

where $s \in \mathbb{C}$ is the Laplace transform indeterminate and $z_2 \in \mathbb{C}$ arises, as before, from the use of the z-transform in the pass-to-pass direction.

III. Stability Theory

The stability theory [6] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a wide range of such processes as special cases, including those described by the state-space models considered in this paper. In terms of their dynamics it is the pass-to-pass coupling (noting again their unique feature) which is critical. This is of the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$ (E_α a Banach space with norm $||\cdot||$) and L_α is a bounded linear operator mapping E_α into itself. (In the case of the processes considered here L_α is a convolution operator.)

For a discrete linear repetitive process of the form considered here, stability along the pass [6] holds if, and only if,

$$\mathcal{C}_{\mathrm{disLRP}}(z_1, z_2) \neq 0 \text{ in } \overline{U}^2$$

where $\overline{U}^2 = \{(z_1, z_2) : |z_1| \geq 1, \ |z_2| \geq 1\}$. The corresponding differential result is

$$\mathcal{C}_{\mathrm{difLRP}}(s, z_2) \neq 0$$
$$\forall \{s, z_2\} \in \{(s, z_2) : \mathrm{Re(s)} \geq 0, \ |\mathrm{z_2}| \leq 1\}$$

The difficulty with these conditions lies in verifying them for a given example since it is necessary to work with functions in two indeterminates. Instead, the following results can be used.

Lemma 1: [7] A discrete linear repetitive process of the form (1) is stable along the pass if, and only if,

i) $\rho(D_0) < 1$
ii) $\rho(A) < 1$
iii) all eigenvalues of $G_{dis}(z_1^{-1}) = C(z_1^{-1}I - A)^{-1}B_0 + D_0, \forall |z_1| = 1$ have modulus strictly less than unity

Lemma 2: [7] A differential linear repetitive process of the form (2) is stable along the pass if, and only if,

i) $\rho(D_0) < 1$
ii) $\sigma(A) \in \mathbb{C}_-$
iii) all eigenvalues of $G_{diff}(s) = C(sI - A)^{-1}B_0 + D_0$ $\forall \omega \geq 0$, have modulus strictly less than unity

Consider condition (i) in both cases. Then this is the necessary and sufficient condition for asymptotic stability, i.e. BIBO stability over the finite pass length. Suppose also that this condition holds and the input sequence applied $\{u_{k+1}\}_k$ converges strongly as $k \to \infty$ (i.e. in the sense of the norm on the underlying function space) to u_∞. Then the strong limit $y_\infty := \lim_{k\to\infty} y_k$ is termed the limit profile corresponding to this input sequence and is described by a 1D linear systems state-space model with state matrix (setting $D = 0$ for simplicity) $A_{lp} := A + B_0(I - D_0)^{-1}C$. Hence under asymptotic stability the process dynamics can, after a sufficiently large number of passes have elapsed, be replaced by those of a 1D (discrete or differential as appropriate) linear systems state-space model. This property does not, however, guarantee that this 1D linear system is stable, i.e. has no growth terms in the along the pass direction. A simple counter-example in the differential case is $A = -1$, $B = 1$, $B_0 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where β is a real scalar. This example is asymptotically stable with resulting limit profile state matrix $A_{lp} = \beta$ and hence the limit profile is unstable for $\beta \geq 0$. Note also that this problem is not avoided by imposing the stability constraint on the matrix A that governs the dynamics produced along any pass with finite k.

In terms of checking the conditions of these two results, the first two conditions in each case are no problem. Also condition (iii) in each case has a Nyquist based interpretation. For SISO examples, this condition requires that the Nyquist plot generated by $G(z_1^{-1})$, respectively $G(s)$, lies inside the unit circle in the complex plane for all $|z_1^{-1}| = 1$, respectively $s = \imath\omega$, $\forall\, \omega$.) Figure 1 illustrates condition (iii) of Lemma 2.

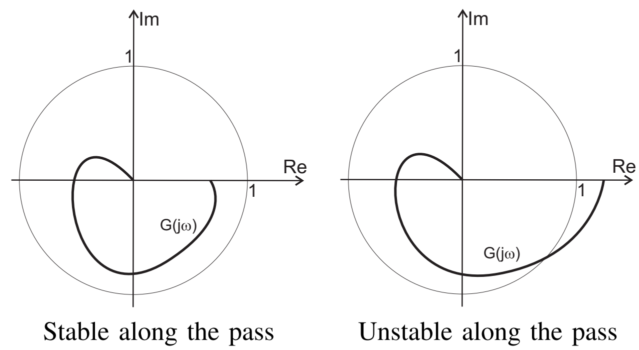


Fig. 1. Graphical representation of condition (iii) of Lemma 1.

One way of avoiding the computational complexity that could arise with condition (iii), and also provide a basis for control law design, is to characterize stability along the pass in terms of a Lyapunov function [6] in each case. These Lyapunov functions must contain contributions from the current pass state and previous pass profile vectors, for example composed of which is the sum of quadratic terms in the current pass state and previous pass profile respectively [6]. In particular for the discrete case consider the Lyapunov function

$$V(k, y) = x_{k+1}(p)^T P_1 x_{k+1}(p) + y_k(p)^T P_2 y_k(p) \quad (5)$$

where $P_i \succ 0$, $i = 1, 2$. Then we have the following result. This leads to LMI conditions that also extend naturally to control law design. The basic results are as follows.

Lemma 3: [6] Assume that there exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that LMI

$$\begin{bmatrix} -P_1 & 0 & P_1 A & P_1 B_0 \\ 0 & -P_2 & P_2 C & P_2 D_0 \\ A^T P_1 & C^T P_2 & -P_1 & 0 \\ B_0^T P_1 & D_0^T P_2 & 0 & -P_2 \end{bmatrix} \prec 0 \quad (6)$$

holds. Then a discrete linear repetitive process of the form (1)) is stable along the pass.

For the differential case, replace p by t. Then the differential process equivalent of (6) is

$$\begin{bmatrix} -P_2 & P_2C & P_2D_0 \\ C^TP_2 & A^TP_1+P_1A & P_1B_0 \\ D_0^TP_2 & B_0^TP_1 & -P_2 \end{bmatrix} \prec 0 \tag{7}$$

IV. Necessary and sufficient stability conditions formulated in terms of LMIs

The LMI based conditions of (6) and (7) are sufficient only and hence can be conservative. In this section, the Kalman-Yakubovich-Popov (KYP) lemma is used as a basis to develop necessary and sufficient conditions for stability along the pass of the SISO versions of the differential and discrete linear repetitive processes considered in this paper. Next we give the required background.

The KYP lemma gives a necessary and sufficient condition for a given transfer-function to satisfy a required frequency domain property over a frequency range in terms of an LMI based condition. Moreover, this lemma can be used to study a specified region in the complex plane by using the following inequality in terms of a given matrix (see - [3] for more details)

$$\begin{bmatrix} G(\delta) & I \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} G^*(\delta) \\ I \end{bmatrix} \prec 0$$

where the symbol δ is used to denote the s or z operator for the differential and discrete cases respectively and

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

is a given matrix that describes the region of interest in the complex plane. Here we require the boundary of the unit circle and the imaginary in the complex plane for the discrete and differential cases respectively. The corresponding choices of Π are

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

respectively.

The KYP lemma has the following form.

Lemma 4: [3] For a given transfer-function $G(\delta)=\mathcal{C}(\delta I-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}$ the following inequality

$$\begin{bmatrix} G(\delta) & I \end{bmatrix} \Pi \begin{bmatrix} G^*(\delta) \\ I \end{bmatrix} \prec 0$$

holds if, and only if, there exist Hermitian matrices P and $Q \succ 0$ such that

$$\begin{bmatrix} \Gamma(P,Q)+\Lambda & \begin{bmatrix} \mathcal{B} \\ \mathcal{D} \end{bmatrix}\Pi_{11} \\ \Pi_{11}[\mathcal{B}^* \ \mathcal{D}^*] & -\Pi_{11} \end{bmatrix} \prec 0$$

where

$$\Gamma(P,Q) = \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ I & 0 \end{bmatrix} \Sigma \begin{bmatrix} \mathcal{A} & \mathcal{C} \\ I & 0 \end{bmatrix}^*$$

$$\Lambda = \begin{bmatrix} 0 & \mathcal{B}^*\Pi_{12} \\ \Pi_{12}^*\mathcal{B} & \mathcal{D}^*\Pi_{12}+\Pi_{12}^*\mathcal{D}+\Pi_{22} \end{bmatrix}$$

and

$$\Sigma = \begin{cases} \begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} & \text{for continuous systems} \\ \begin{bmatrix} P & -Q \\ -Q & -P+2Q \end{bmatrix} & \text{for discrete systems} \end{cases}$$

A. Application to Discrete Linear Repetitive Processes

We require the following preliminary result.

Theorem 1: Consider a SISO controllable and observable 1D discrete linear system described by the state-space model

$$\begin{aligned} x(p+1) &= \mathcal{A}x(p)+\mathcal{B}u(p) \\ y(p) &= \mathcal{C}x(p)+\mathcal{D}u(p) \end{aligned}$$

with corresponding transfer-function $G(z) = \mathcal{C}(Iz - \mathcal{A})^{-1}\mathcal{B}+\mathcal{D}$. Then the following two conditions are equivalent

i) $|G(z)| < 1, \forall z = e^{j\omega},\ \omega \in [0, 2\pi]$
ii) there exist $Q \succ 0$ and a symmetric matrix P such that

$$\begin{bmatrix} \mathcal{A}P\mathcal{A}^T-P-Q\mathcal{A}^T-\mathcal{A}Q+2Q & \mathcal{A}Q\mathcal{C}^T+P\mathcal{C}^T & \mathcal{B} \\ \mathcal{C}P\mathcal{A}^T-\mathcal{C}Q & \mathcal{C}P\mathcal{C}^T-I & \mathcal{D} \\ \mathcal{B}^T & \mathcal{D}^T & -I \end{bmatrix} \prec 0 \tag{9}$$

Proof: First note that (i) is equivalent to the requirement that the Nyquist plot $G(z)$ lies in the interior of the of the unit circle in the complex plane and is the necessary and sufficient condition for asymptotic stability of SISO discrete linear systems. Also the interior of the unit circle, i.e. the stability region can (see (8)) be written as

$$\begin{bmatrix} G & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} G^* \\ I \end{bmatrix} = |G|^2 - 1 < 0$$

This is convex and hence can be described in terms of LMIs. Hence [3] condition (i) above is equivalent to the LMI condition (9). ■

Now we can use Lemma 1 to obtain LMI based necessary and sufficient conditions for stability along the pass of SISO discrete linear repetitive processes of the form considered here.

Theorem 2: A SISO discrete linear repetitive process of the form (1) (where the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ observable) is stable along the pass if, and only if, there exist $R \succ 0$, $S \succ 0$, $Q \succ 0$ and a symmetric matrix P such that the following LMIs are feasible

i) $D_0^TRD_0 - R \prec 0$
ii) $A^TSA - S \prec 0$
iii)

$$\begin{bmatrix} APA^T-P-QA^T-AQ+2Q & APC^T-QC^T & B_0 \\ CPA^T-CQ & CPC^T-I & D_0 \\ B_0^T & D_0^T & -I \end{bmatrix} \prec 0 \tag{10}$$

Proof: The first two conditions follow immediately from Lyapunov stability theory for discrete linear systems. The third LMI follows immediately on applying Lemma 1 to the transfer-function $G_{dis}(z_1^{-1})$. ■

B. Application to Differential Linear Repetitive Processes

We require the following preliminary result.

Lemma 5: Consider a SISO controllable and observable 1D differential linear system described by the state-space model

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$$
$$y(t) = \mathcal{C}x(t) + \mathcal{D}u(t)$$

with corresponding transfer-function $G(s) = \mathcal{C}(Is - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$. Then the following two conditions are equivalent

i) $|G(j\omega)| < 1, \forall \omega \in \mathbb{R}$

ii) there exist $Q \succ 0$ and a symmetric matrix P such that

$$\begin{bmatrix} \mathcal{A}Q\mathcal{A}^T + P\mathcal{A}^T + \mathcal{A}P & \mathcal{A}Q\mathcal{C}^T + P\mathcal{C}^T & \mathcal{B} \\ \mathcal{C}Q\mathcal{A}^T + \mathcal{C}P & \mathcal{C}Q\mathcal{C}^T - I & \mathcal{D} \\ \mathcal{B}^T & \mathcal{D}^T & -I \end{bmatrix} \prec 0 \quad (11)$$

Proof: This follows, with routine replacement of the boundary of the unit circle by the imaginary axis in the complex plane, that for Lemma 1 and hence the details are omitted. ■

Now we can use Lemma 5 to obtain LMI based necessary and sufficient conditions for stability along the pass of SISO differential linear repetitive processes of the form considered here.

Theorem 3: A SISO differential linear repetitive process of the form (2) is stable along the pass if, and only if, there exist $Q \succ 0$, $R \succ 0$, $X \succ 0$ and a symmetric matrix P such that the following LMIs are feasible

i) $D_0^T R D_0 - R \prec 0$

ii) $A^T X + XA \prec 0$

iii)

$$\begin{bmatrix} AQA^T + PA^T + AP & AQC^T + PC^T & B_0 \\ CQA^T + CP & CQC^T - I & D_0 \\ B_0^T & D_0^T & -I \end{bmatrix} \prec 0 \quad (12)$$

Proof: The first two conditions follow immediately from Lyapunov stability theory for 1D discrete and differential linear systems respectively. The third LMI follows immediately on applying Lemma 1 to the transfer-function $G_{diff}(s)$. ■

V. Illustrative Examples

In this section, we give two examples to demonstrate the effectiveness of the results developed.

Example 1: Consider the discrete linear repetitive process defined by the state-space model matrices

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & -0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1.0 \\ 0.1 \end{bmatrix}$$
$$C = \begin{bmatrix} -0.2 & 0.6 \end{bmatrix}, \quad D_0 = -0.7 \quad (13)$$

Here

$$\sigma(A) = \{0.5742, -0.1742\}, \quad \sigma(D_0) = -0.7$$

and it remains to check condition (iii) which in fact holds and hence this example is stable along the pass. Note however that the sufficient but not necessary condition for stability along the pass given by the LMI (6) is inconclusive in this case. The Nyquist plot in Figure 2 confirms that this example is indeed stable along the pass.

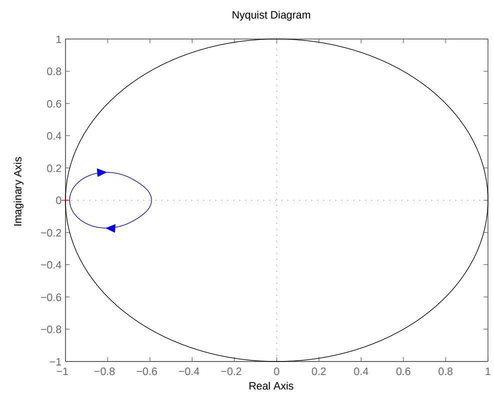


Fig. 2. Nyquist plot of the stable along the pass discrete process

Example 2: Consider the discrete linear repetitive process defined by the state-space model matrices

$$A = \begin{bmatrix} -2.929 & -0.3186 \\ -0.3186 & -0.8829 \end{bmatrix}, \quad B_0 = \begin{bmatrix} -0.2 \\ -1.50 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.9 & 1.2 \end{bmatrix}, \quad D_0 = 0.99 \quad (14)$$

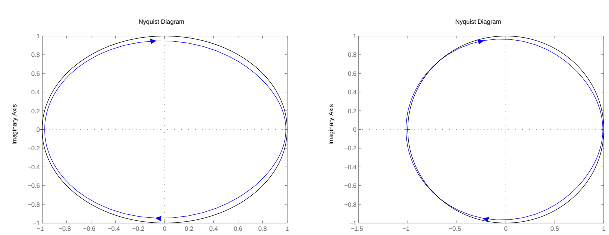
It is easy to check that two first conditions of Theorem 3 are satisfied since

$$\sigma(D_0) = 0.99, \quad \sigma(A) = \{-2.9775, -0.8344\}$$

Also the LMI of (12) is feasible and one solution is

$$Q = \begin{bmatrix} 0.1877 & -0.1307 \\ -0.1307 & 0.0987 \end{bmatrix}, \quad P = \begin{bmatrix} 0.4735 & -0.1510 \\ -0.1510 & 1.3623 \end{bmatrix}$$

Hence the example defined by (14) is stable along the pass as confirmed by the Nyquist plot of Figure 3(a). If, however,



(a) Stable along the pass process (b) Unstable along the pass proces

Fig. 3. Nyquist plots

$B_0 = [-0.2 \quad -1.52]^T$ (which has no influence on two first conditions) we see that the resulting process is unstable along the pass since the LMI of (12) has no solution — see also Figure 3(b).

VI. Application to ILC

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive (or pass-to-pass) mode with the requirement that a reference trajectory $y_{ref}(t)$ defined over a finite interval $0 \leq t \leq \alpha$ is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task, chemical batch processes or, more generally, the class of tracking systems.

In ILC, a major objective is to achieve convergence of the trial-to-trial error and often this has been treated as the only one that needs to be considered. In fact, it is possible that enforcing fast convergence could lead to unsatisfactory performance along the trial. One way of preventing this is to exploit the fact that ILC schemes can be modeled as linear repetitive processes and design the scheme to ensure stability along the pass. Previous work [2] has shown how this leads to LMI based design with experimental validation on a three axis gantry robot where each of them is controlled individually, i.e. there SISO control design problems.

The only difficulty with this previous work is that the design is based on sufficient but not necessary conditions and hence conservativeness. An alternative would be to apply the necessary and sufficient LMI based conditions developed in this paper where the result below gives the stability along the pass condition for one ILC law.

We work in the discrete domain and so assume that the process dynamics have been sampled by the zero-order hold method at a uniform rate T_s seconds to produce a discrete state-space model with matrices $\{A, B, C\}$. Also introduce

$$\begin{aligned} \eta_{k+1}(p+1) &= x_{k+1}(p) - x_k(p) \\ \Delta u_{k+1}(p) &= u_{k+1}(p) - u_k(p) \end{aligned}$$

and let $e_k(p) = y_{ref}(p) - y_k(p)$ denote the current pass error, where $y_{ref}(p)$ is the pre-specified reference signal and $\Delta u_{k+1}(p)$ is the change in the control signal between two successive passes. Then it is possible to proceed as in [2] and use an ILC law which requires the current trial state vector $x_k(p)$ of the plant using

$$\Delta u_{k+1}(p) = K_1 \eta_{k+1}(p+1) + K_2 e_k(p+1)$$

and hence the controlled system dynamics can be written as

$$\begin{aligned} \eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \hat{B}_0 e_k(p) \\ e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \hat{D}_0 e_k(p) \end{aligned} \qquad (15)$$

where

$$\begin{aligned} &\hat{A} = A + BK_1, \ \hat{B}_0 = BK_2, \\ &\hat{C} = -C(A+BK_1), \hat{D}_0 = (I - CBK_2) \end{aligned}$$

The following result now gives necessary and sufficient conditions for stability along the pass in terms of matrix inequality conditions.

Theorem 4: A SISO discrete linear repetitive process of the form (15) is stable along the pass if, and only if, there exist $\bar{r} \succ 0$, $\bar{S} \succ 0$, $Q \succ 0$ and a symmetric matrix P such that the following matrix inequalities are feasible

i) $\hat{D}_0^T \bar{r} \hat{D}_0 - \bar{r} \prec 0$
ii) $\hat{A}^T \bar{S} \hat{A} - \bar{S} \prec 0$
iii)

$$\begin{bmatrix} \hat{A}P\hat{A}^T - P - Q\hat{A}^T - \hat{A}Q + 2Q & \hat{A}P\hat{C}^T - Q\hat{C}^T & \hat{B}_0 \\ \hat{C}P\hat{A}^T - \hat{C}Q & \hat{C}P\hat{C}^T - I & \hat{D}_0 \\ \hat{B}_0^T & \hat{D}_0^T & -I \end{bmatrix} \prec 0$$

With further research this should lead to necessary and sufficient algorithms for control law design.

VII. Conclusions

This paper has developed necessary and sufficient conditions for stability along the pass of both differential and discrete linear repetitive processes in the form of LMI based conditions using a dissipative setting. Previous work had only led to sufficient but not necessary conditions and hence the possibility of conservative answers especially when considering control law design. Obvious areas for further work include extending these results to control law design in the case when the matrix P in the control law design result here is not positive definite, leading to bilinear terms in the form od a product of the matrix P and the control law matrices. This means that the resulting inequalities are no longer linear and therefore non-convex. One of possible method to overcome this problem is to apply the Youla parametrization. In the ILC and other applications area there is also much to be done beyond extending the last result here to allow for control law design. One strong feature of the repetitive process setting for ILC analysis is that it facilitates control law design for pass-to-pass (or trial-to-trial) error convergence and performance along the pass. Further research is clearly required in order to fully develop this aspect.

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