Practical stability of perturbed event-driven controlled linear systems

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***Abstract*— Many plants are regulated by digital controllers that run at a constant sampling frequency, thereby requiring a high processor load for the computations. To guarantee a good control performance, such a high sampling frequency might be required at some periods of time, but not necessarily continuously. By using an event-driven control scheme that triggers the update of the control value only when the (tracking or stabilization) error is large, the average processor load can be reduced considerably. Although event-driven control can be effective from a CPU-load perspective, the analysis of such control schemes is much more involved than that of conventional schemes and is a widely open research area. This paper investigates the control performance of an event-driven controlled continuous-time linear system with additive disturbances in terms of practical stability (ultimate boundedness). By using the derived results, the event-driven controller can be tuned to get satisfactorily transient behavior and desirable ultimate bounds, while reducing the required average processor load for its implementation. Several examples illustrate the theory.**

***Index Terms*— Practical stability, sampled-data control, processor load, ultimate boundedness, robust invariance, piecewise linear systems.**

I. Introduction

Many plants are regulated by digital controllers that run at a constant (relatively high) sampling frequency, thereby requiring a high processor load for the computations. To achieve accurate control, controllers require a high sampling frequency at certain periods of time, but do not require this at each interval of time. This opens up the possibility to lower the average processor load needed for the implementation of the controller. In the literature [1], [2], [7], [14] event-driven control strategies have been proposed to create a negotiable environment to make such a compromise between processor load and control performance. However, theoretical analysis of the proposed event-driven controllers is lacking in literature. Although the event-driven controllers considered here are less complicated in comparison with the cited work, this work provides the first step in a proper analysis of these types of control loops.

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To show the potential of reducing the involved control computations without deteriorating the control performance significantly, consider the following simple continuous-time plant

$$\dot{x}(t) = 0.5x(t) + 10u(t) + 3w(t) \tag{1}$$

with $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}$ the state, control input and disturbance at time $t \in \mathbb{R}_+$, respectively. The additive disturbance satisfies $-10 \leq w(t) \leq 10$. This system will be controlled by a discrete-time controller

$$u_k = \begin{cases} -0.45x_k, & \text{if } |x_k| \geq e_T \\ u_{k-1}, & \text{if } |x_k| < e_T, \end{cases} \tag{2}$$

that runs at a fixed sample time of $T_s = 0.1$ time units. Here, e_T denotes a parameter that determines the region $\mathcal{B} := \{x \in \mathbb{R} \mid |x| < e_T\}$ close to the origin in which the control values are not updated. Note that outside $\mathcal{B}$ the control values are updated in an "normal fashion." This particular situation is referred to as uniform sampling. We will also consider the non-uniform case where reaching the boundary of $\mathcal{B}$ will be the event trigger - in addition to a fixed update rate outside $\mathcal{B}$ - for updating the control values. Figure 1 displays the ratio of the number of control updates in comparison to the case where the updates are performed each sample time (i.e. $u_k = -0.45x_k$ for all x_k) and the maximal value of the state variable (after transients) $x_{max} := \limsup_{t\to\infty} |x(t)|$, respectively, versus the parameter e_T. The results are based on simulations. One sees that by relaxing the control accuracy (in terms of the ultimate bound $x_{\max}$ on the state) one can reduce almost 80% of the control computations. Depending on the ratio between the computational complexity of the control algorithm, the overhead of the event triggering mechanisms and i/o access

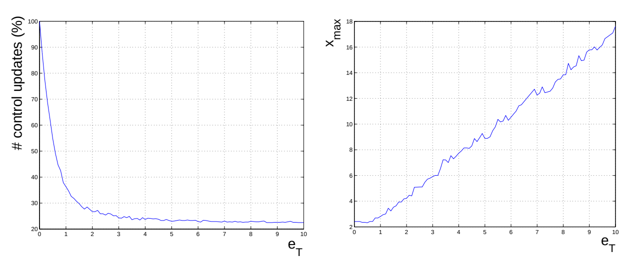

Fig. 1. e_T versus the control effort and x_{max} for system (1)-(2).

of the processor, the reduction of control computation indeed lowers the processor load considerably. Initial experimental measurements [15] show promising results.

It is of interest to investigate how to choose the controller gain and e_T (or more general $\mathcal{B}$) in order to get desirable closed-loop behavior on one hand and low processor usage on the other. Note that in this paper we both select the controller parameters and the way the events are generated that determine when the control values are updated. This is in contrast with the effect of uncertain and time-varying delays ("jitter") introduced in the loop due to the real-time implementation of control algorithms in embedded systems. In that case the variations in the "event triggering" can be considered as a disturbance and one designs compensators that are robust to it, see e.g. [3], [11].

This paper provides theory and insight to understand and tune event-driven controlled linear systems for both the uniform and the non-uniform case. The performance of these novel control strategies is addressed in terms of ultimate boundedness (practical stability), robustly positively invariant sets, and guaranteed speed of convergence [4]. Depending on the particular event triggers for updating the control values, properties like robust positive invariance or convergence to a set for the *perturbed event-driven linear system* can be derived either from a *perturbed discrete-time linear system* or from a *perturbed discrete-time (non-deterministic) piecewise linear (PWL) system*. Since results for robust invariance and ultimate boundedness are known for discrete-time linear systems, see e.g. [4], [5], [8], [9], [12], and piecewise linear systems, see e.g. [13], [10], these results can be carried over to event-driven controlled systems. In this way we can examine how the tuning parameters of the controller should be chosen to obtain satisfactory control performance on one hand and computational effort of its implementation on the other.

II. Preliminaries

A set $\Omega \in \mathbb{R}^n$ is a C-set, if it is compact, convex and contains 0 in its interior. For a set Ω we denote its interior, its closure and its boundary by intΩ, clΩ and $\partial\Omega$, respectively. We define the Minkowski functional Φ_Ω for a C-set Ω as $\Phi_\Omega(x) := \inf\{\lambda > 0 \mid x \in \lambda\Omega\}$. Note that $x \in \Omega$ if and only if $\Phi_\Omega(x) \leq 1$. The symbol $\oplus$ denotes the Minkowski sum of two sets: $\mathcal{U} \oplus \mathcal{V} := \{u + v \mid u \in \mathcal{U}, v \in \mathcal{V}\}$. For two sets Ω_1 and Ω_2 of $\mathbb{R}^n$, we denote the set difference $\Omega_1 \setminus \Omega_2$ is defined as $\{x \in \Omega_1 \mid x \notin \Omega_2\}$. The complement of $\Omega \subset \mathbb{R}^n$ is defined as $\mathbb{R}^n \setminus \Omega$ and is denoted by Ω^c.

Consider a continuous-time system

$$\dot{x}(t) = f(t, x(t), w(t)) \tag{3}$$

with $x(t) \in \mathbb{R}^n$ the state variable and $w(t) \in \mathcal{W}_c$ the disturbance at time $t \in \mathbb{R}_+$ or a discrete-time difference equation

$$x_{k+1} = f(k, x_k, w_k) \tag{4}$$

with $x_k \in \mathbb{R}^n$ the state and $w_k \in \mathcal{W}_d$ the disturbance at discrete-time $k \in \mathbb{N}$. $\mathcal{W}_c$ and $\mathcal{W}_d$ denote the disturbance sets, which are assumed to be convex, compact and contain 0. We define the set $\mathcal{L}_1([0, T_s] \mapsto \mathbb{R}^p)$ as the Lebesgue space of integrable functions on $[0, T_s]$ to $\mathbb{R}^p$ and $\mathcal{L}_1^{\text{loc}}([0, \infty) \mapsto \mathbb{R}^p)$ as the Lebesgue space of locally integrable functions from $[0, \infty)$ to $\mathbb{R}^p$.

Definition 2.1: Given $0 \leq \lambda \leq 1$. The set Ω is a (robustly) λ-contractive set for the discrete-time difference equation (4), if for any $x \in \Omega$, $k \in \mathbb{N}$ and any $w \in \mathcal{W}_d$ it holds that $f(k, x, w) \in \lambda\Omega$. For $\lambda = 1$ we say that Ω is robustly positively invariant (RPI).

Definition 2.2: [4] We call the discrete-time difference equation (4) *ultimately bounded* (UB) to the set Ω, if for each $x_0 \in \mathbb{R}^n$ there exists a $K(x_0) > 0$ such that any state trajectory of (4) with initial condition x_0 (and any arbitrary realization of the disturbance $w : \mathbb{N} \mapsto \mathcal{W}_d$) satisfies $x_k \in \Omega$ for all $k \geq K(x_0)$. Similarly, we call (3) *ultimately bounded* (UB) to the set Ω, if for every initial condition $x(0) \in \mathbb{R}^n$ there exists a $T(x(0)) > 0$ such that any state trajectory of (3) with initial condition $x(0)$ (and any arbitrary realization of the disturbance $w :\in \mathcal{L}_1^{\text{loc}}([0, \infty) \mapsto \mathbb{R}^p)$ with $w(t) \in \mathcal{W}_c$ a.e.) satisfies $x(t) \in \Omega$ for all $t \geq T(x(0))$.

Definition 2.3: We say that the system (4) has a *convergence index* $0 \leq \lambda \leq 1$ to the C-set Ω, if (4) is UB to Ω and there exists a C-set $\mathcal{S} \subseteq \Omega$ such that $\Phi_\mathcal{S}(x_{k+1}) \leq \lambda\Phi_\mathcal{S}(x_k)$ for all $k \in \mathbb{N}$, $x_k \notin$ intΩ and all $w_k \in \mathcal{W}_d$ where $x_{k+1} = f(k, x_k, w_k)$.

Note that this is a minor adaptation of the definition in [4] for which the latter condition should hold for any $x_k \notin$ int$\mathcal{S}$, which is a more stringent condition.

III. Problem formulation

We consider the system described by

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c w(t), \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the control input and $w(t) \in \mathcal{W}_c$ the unknown disturbance, respectively, at time $t \in \mathbb{R}_+$. $\mathcal{W}_c \subset \mathbb{R}^p$ is a convex and compact set, which contains the origin. $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$ and $E_c \in \mathbb{R}^{n\times p}$ are constant matrices.

The system will be controlled by a discrete-time state-feedback controller with gain $F \in \mathbb{R}^{m\times n}$, i.e.

$$u_k = F x_k, \tag{6}$$

where $x_k = x(\tau_k)$, $u_k = u(\tau_k)$ using the zero-order hold $u(t) = u_k$ for all $t \in [\tau_k, \tau_{k+1})$.

Normally, the *event times* τ_k are related through $\tau_{k+1} = \tau_k + T_s$, where T_s is a fixed sample time meaning that the control value is updated every T_s time units according to (6). To reduce the number of required control calculations, in this paper we propose not to update the control value if the state $x(\tau_k)$ is contained in a set $\mathcal{B}$ close to the origin. The consequences for the control performance in terms of control accuracy (ultimate bounds) and speed of convergence will be investigated. As such, we consider a set $\mathcal{B}$ that is open[1] and

[1] This is merely a technical condition to make the following exposition more compact and clear. This is not a restrictive condition.

contains the origin. If the state of the system is in $\mathcal{B}$ at the *event times* τ_k, the controller output will not be calculated and updated. If the state is outside $\mathcal{B}$, an update is performed according to (6). Hence, the closed-loop system (5)-(6) is modified to

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c w(t)$$
$$u(t) = \begin{cases} Fx(\tau_k) & \text{if } x(\tau_k) \notin \mathcal{B} \\ u(\tau_{k-1}) & \text{if } x(\tau_k) \in \mathcal{B} \end{cases} \quad \text{for } t \in [\tau_k, \tau_{k+1}), \tag{7}$$

where we still have to specify how the event times τ_k are generated. We consider two ways of updating the event times τ_k: a *non-uniform* (triggered by the event of leaving $\mathcal{B}$) and a *uniform* (sampling at a fixed sample time T_s) manner. Note that the system (7) together with a particular way of generating the event times can be considered as a discrete-time system, if one restricts oneself to the event times. Hence, this means that the definitions for (4) in section II can easily be generlazed to apply for (7) "on the event times."

A. Problem 1: non-uniform sampling

The *event times* τ_k are chosen such that

$$\begin{array}{lll} \tau_{k+1} = \tau_k + T_s & \text{if} & x(\tau_k) \notin \mathcal{B} \\ \tau_{k+1} = \tau_{exit} & \text{if} & x(\tau_k) \in \mathcal{B}, \end{array} \tag{8}$$

where $\tau_{exit} = \inf\{t > \tau_k \mid x(t) \notin \mathcal{B}\}$ is the time instant at which $x(t)$ exits $\mathcal{B}$ (after being inside $\mathcal{B}$ at the previous event time). For the situation in which $x(0) \in \mathcal{B}$ ($\tau_0 = 0$) we assume that $u_{\tau_0} = 0$.

From (7) it can be seen that the control updates are not synchronous. The duration that the state of the system remains inside $\mathcal{B}$ causes asynchronicity, although T_s is a fixed sample time outside $\mathcal{B}$.

B. Problem 2: uniform sampling

In the previous section the control strategy is such that the control value is updated as soon as the boundary of $\mathcal{B}$ is hit and the state was inside $\mathcal{B}$ at the previous event time. In many applications such implementation would not be optimal with respect to the scheduling of tasks on a processor. Whether the state of the system is inside or outside $\mathcal{B}$ will often be detected on a constant rate. If this rate is chosen equal to the sampling rate of the controller, i.e. having a period time T_s, the system description (7) can be used with the *uniformly* distributed event times τ_k, $k = 0, 1, 2, \ldots$ with $\tau_0 = 0$ and

$$\tau_{k+1} = \tau_k + T_s \tag{9}$$

C. Control objectives

The control objective is a "stabilization problem" in the sense of controlling the state towards a region Ω close to the origin and keeping it there, as we cannot expect asymptotic stability due to the type of control strategy employed and the presence of disturbances. Hence, this means that we consider *practical stability* which has been used widely to prove system performance in the context of non-linear and perturbed systems. A term that is also often used in this context is uniform *ultimate boundedness* [4].

Problem 3.1: Let a desired ultimate bound $\Omega \subset \mathbb{R}^n$ containing 0 in the interior be given and let $0 \leq \lambda \leq 1$ be a desirable convergence index. Construct F and $\mathcal{B}$ such that the system (7) with the event times given by either (8) or (9) is UB to Ω (as a continuous-time system) and (7) has a convergence index λ towards Ω (as a discrete-time system on the event times τ_k).

IV. General Theory

Problem 3.1 will be solved in two stages. First properties on UB to Ω and convergence indices to Ω are obtained for the event-driven system (7) *on the event times*. Next bounds on the intersample behavior (see Section IV-C below) will be derived that enlarge Ω to $\tilde{\Omega}$ such that the ultimate bound $\tilde{\Omega}$ is guaranteed for all (continuous) times t.

To do so, the discrete-time system

$$x_{k+1} = (A + BF)x_k + w_k = A_{cl}x_k + w_k \text{ with} \tag{10}$$

$$\begin{array}{l} A := e^{A_c T_s} \\ B := \int_0^{T_s} e^{A_c\theta} d\theta B_c \\ w_k := \int_{\tau_k}^{\tau_{k+1}} e^{A_c(\tau_{k+1}-\theta)} E_c w(\theta) d\theta \\ A_{cl} := A + BF \end{array} \tag{11}$$

will play an important role in the analysis. Indeed, for both the uniform and non-uniform sampling case, the system behaves far away from the set $\mathcal{B}$ (at the event times) as (10). We use the shorthand notation $x(\tau_k) = x_k$ here. Note that this system is only representing the system (7) at the event times, when $x(\tau_k) \notin \mathcal{B}$. The bounds on $w(t)$ given by $\mathcal{W}_c$ are transformed into bounds on w_k given by $\mathcal{W}_d := \{\int_0^{T_s} e^{A_c(T_s-\theta)} E_c w(\theta) d\theta \mid w \in \mathcal{L}_1([0, T_s] \mapsto \mathbb{R}^p),\ w(t) \in \mathcal{W}_c \text{ a.e. }\}$. Since $\mathcal{W}_c$ is convex, compact and contains 0, $\mathcal{W}_d$ is convex, compact and contains 0.

A. Non-uniform sampling

As we will see in the theorem below, ultimate bounds for the *linear discrete-time system* (10) can be used to find ultimate bounds for the *event-driven system* (7) with non-uniform sampling (8).

Theorem 4.1: Consider the system (7)-(8) with $\mathcal{W}_c$ a closed, convex set containing 0, F given and $\mathcal{B}$ an open set containing the origin.

1) If Ω is a RPI set for the *linear discrete-time system* (10) with disturbances in $\mathcal{W}_d$ and cl$\mathcal{B} \subseteq \Omega$, then Ω is a RPI set for the *event-driven system* (7)-(8) *on the event times*, meaning that if $x_0 \in \Omega$, then $x_{x_0,w}(\tau_k) \in \Omega$ where $x_{x_0,w}(\cdot)$ denotes the solution to (7)-(8) with $x(0) = x_0$ and the realization of the disturbance given by $w :\in \mathcal{L}_1^{\text{loc}}([0, \infty) \mapsto \mathcal{W}_c)$.
2) If the *linear discrete-time system* (10) with disturbances in $\mathcal{W}_d$ is UB to the RPI set Ω and cl$\mathcal{B} \subseteq \Omega$, then the *event-driven system* (7)-(8) *on the event times* is UB to Ω.
3) If the *linear discrete-time system* (10) with disturbances in $\mathcal{W}_d$ has convergence factor $\lambda \leq 1$ to the RPI C-set Ω and cl$\mathcal{B} \subseteq \Omega$, then the *event-driven system*

(7)-(8) *on the event times* has convergence index λ to Ω.

Proof: 1) Let $x(\tau_k) \in \Omega$. Then we can distinguish two cases: If $x(\tau_k) \in \mathcal{B}$, we will either remain in $\mathcal{B}$ forever (thereby not destroying robust positive invariance) or an exit time τ_{k+1} will occur for which $x(\tau_{k+1}) \in \partial\mathcal{B} \subset \text{cl}\mathcal{B} \subseteq \Omega$. The other case is that $x(\tau_k) \notin \mathcal{B}$, then $\tau_{k+1} = \tau_k + T_s$ according to (8) and the update of the state over the interval $[\tau_k, \tau_{k+1}]$ is governed by (10) for some $w_k \in \mathcal{W}_d$. As Ω is a RPI set for (10), this means that $x(\tau_{k+1}) \in \Omega$ (irrespective of the realization of the noise). Hence, we proved that if $x(\tau_k) \in \Omega$ then $x(\tau_{k+1}) \in \Omega$ meaning that Ω is RPI for the event-driven system at the event times.

2) If $x(0) \in \Omega$, then due to RPI of Ω the system (7)-(8) stays within Ω on the event times as outlined in the first part of the proof. If $x(0) \notin \Omega$ and thus $x(0) \notin \mathcal{B}$, the system is governed by (10) on the event times as long as $x(\tau_k) \notin \Omega$. Since (10) is UB to Ω there exists a time $K(x(0))$ such that $x(\tau_{K(x(0))}) \in \Omega$. Since Ω is RPI for (7)-(8) on the event times, we have $x(\tau_k) \in \Omega$ for all $k \geq K(x(0))$. This completes the proof of statement 2.

3) Similar reasoning applies to the system (7)-(8) to have a convergence index λ to the set Ω. ■

B. Uniform sampling

As mentioned before, the non-uniform update scheme is hard to implement in practice. Uniform sampling might be more relevant from a practical point of view. However, in contrast to non-uniform sampling the properties of the discrete-time linear system *do not* transfer to the event-driven system in this case. As we will see, we will need a piecewise linear (PWL) model to analyse the event-driven systems using uniform sampling. For ease of exposition and brevity, we present only the *unperturbed* case here.

To be able to compute an ultimate bound that solves problem 3.1, we consider (7) with the uniform event times as in (9), i.e. $\tau_{k+1} = \tau_k + T_s$ with $\tau_0 = 0$ and we take $x_k := x(\tau_k)$, $k = 0, 1, 2, \ldots$. At the event times the system is described by the discrete-time system

$$x_{k+1} = Ax_k + Bu_k \qquad u_k = \begin{cases} Fx_k & \text{if } x_k \notin \mathcal{B} \\ u_{k-1} & \text{if } x_k \in \mathcal{B}. \end{cases} \tag{12}$$

Since we only sample the system at uniformly distributed times τ_k, we do not know how far we are outside $\mathcal{B}$ before we detect that the state left $\mathcal{B}$. The reason is that in $\mathcal{B}$ the control value is held and depends on the state on the event time just before $\mathcal{B}$ was entered (possibly several event times ago), which prevents that the analysis can be based on (10) as in the non-uniform case. Instead we explicitly are going to compute the maps that relate the state just after entering $\mathcal{B}$ to the state after just leaving $\mathcal{B}$ again. Depending on how long the control value is held, a different map defines the update that relates both states. It will turn out that in this way a piecewise linear (PWL) model is obtained in which we abstract away from the time that the system is inside $\mathcal{B}$. Using this *PWL system* properties related to UB and convergence factors can be translated to the original system (7)-(9) on the event times. We will start with presenting how this PWL description can be created using the following assumption.

Assumption 4.1: $A + BF$ is non-singular.

To define the map g_p for the different periods of time (denoted by p) that the state stays in $\mathcal{B}$, we consider first the case $p = 0$, i.e. $x_k \notin \mathcal{B}$ the system update matrix is given by

$$x_{k+1} := g_0(x_k) = (A + BF)x_k. \tag{13}$$

For $p = 1$ we assume that $x_{k-1} \notin \mathcal{B}$, $x_k \in \mathcal{B}$ and then $x_{k+1} \notin \mathcal{B}$. The function g_1 defines the mapping from x_k to x_{k+1} in this case. This update of the state is given by $x_{k+1} = Ax_k + Bu_k$ with $u_k = u_{k-1} = Fx_{k-1}$ (since the control value is held). From (12) we have that $x_k = (A+BF)x_{k-1}$ and thus $x_{k-1} = (A + BF)^{-1}x_k$. This gives

$$u_k = u_{k-1} = F(A + BF)^{-1}x_k. \tag{14}$$

Hence,

$$\begin{aligned} x_{k+1} &= g_1(x_k) \\ &:= Ax_k + BF(A + BF)^{-1}x_k \end{aligned} \tag{15}$$

Similarly, suppose we stay p steps in $\mathcal{B}$ before leaving $\mathcal{B}$ again (i.e. $x_{k-1} \notin \mathcal{B}$, then $x_k \in \mathcal{B}$, $x_{k+1} \in \mathcal{B}$, ..., $x_{k+p-1} \in \mathcal{B}$ and then $x_{k+p} \notin \mathcal{B}$). We obtain the function g_p that maps x_k to x_{k+p} as follows by using repetitively

$$x_{k+i} = Ax_{k+i-1} + Bu_{k+i-1}, \tag{16}$$

for $i = 1, ..., p$. Since the control value is held, it holds that $u_{k+p-1} = u_{k+p-2} = ... = u_k = u_{k-1}$. As $u_{k-1} = F(A+BF)^{-1}x_k$ by (14), we can express x_{k+p} as a function of x_k:

$$\begin{aligned} x_{k+p} &= g_p(x_k) := A^p x_k + \\ &+ [A^{p-1} + A^{p-2} + \ldots + I]BF(A + BF)^{-1}x_k. \end{aligned} \tag{17}$$

Now that the maps g_p are defined, the region D_p has to be determined for which the map g_p is active. For $p = 0$ this is straightforward as $D_0 := \mathcal{B}^c$, which denotes the complement of $\mathcal{B}$. For $p > 0$ D_p is given by those x_k for which there exists an $x_{k-1} \notin \mathcal{B}$ such that $x_k = (A + BF)x_{k-1} \in \mathcal{B}$, $x_{k+1} \in \mathcal{B}$, ..., $x_{k+p-1} \in \mathcal{B}$ and then $x_{k+p} \notin \mathcal{B}$ is satisfied. Hence, for $p = 1, 2, \ldots$ we have

$$\begin{aligned} D_p \;:=\; &\{x \in \mathcal{B} \mid (A + BF)^{-1}x \notin \mathcal{B} \text{ and} \\ &g_j(x) \in \mathcal{B} \text{ for } j = 1, \ldots, p-1 \\ &\text{and } g_p(x) \notin \mathcal{B}\}. \end{aligned} \tag{18}$$

We also define the set of states that remain inside $\mathcal{B}$ forever after entering it from outside $\mathcal{B}$.

$$\begin{aligned} D_\infty \;:=\; &\{x \in \mathcal{B} \mid (A + BF)^{-1}x \notin \mathcal{B} \\ &g_j(x) \in \mathcal{B} \text{ for all } j = 1, 2, \ldots\}. \end{aligned} \tag{19}$$

Note that $D_i \cap D_j = \emptyset$ if $i \neq j$.

Finally, we introduce the set R_B which contains all possible values of x_k within $\mathcal{B}$, that can be reached within one discrete time-step starting from a state x_{k-1} outside $\mathcal{B}$:

$$R_B := \{x \in \mathcal{B} \mid (A + BF)^{-1}x \notin \mathcal{B}\}.$$

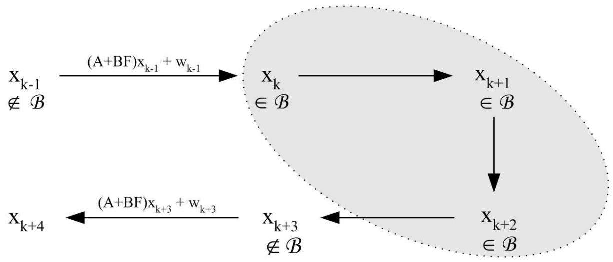

Fig. 2. State transformations for k.

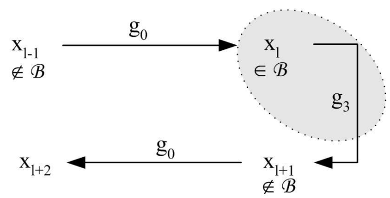

Fig. 3. State transformations for l.

Note that it holds that

$$R_B = D_\infty \cup \bigcup_{i=1}^{\infty} D_i.$$

To obtain a finite representation of the piecewise linear system, we need the existence of a p_{max} such that

$$R_B = D_\infty \cup \bigcup_{i=1}^{p_{max}} D_i. \tag{20}$$

Remark 4.1: Deriving conditions for which the existence of such a finite p_{max} is guaranteed is an open issue. One of the complications is for instance that $D_i = \emptyset$ does not necessarily imply that $D_{i+1} = \emptyset$. Also the computation of D_∞ is not straightforward. However, a condition that guarantees the emptiness of D_∞ and the existence of a p_{max} such that (20) holds, is, for instance, that all the eigenvalues of the matrix A lie outside the closed unit circle of the complex half plane and $A+BF$ does not have an eigenvalue 1 (which is typically the case as $A+BF$ is chosen such that all eigenvalues are inside the open unit circle). One can even compute an explicit upperbound for p_{max}. A kind of "reverse" Lyapunov argument proves this statement. Future research will be focussed on this matter.

In figures 2 and 3 it is illustrated how we abstract away from the motion inside $\mathcal{B}$. The iteration parameter k is substituted for l after abstracting away from the motion of the system's state inside $\mathcal{B}$. This notation will be maintained for the rest of this paper. Therefore, we replace $x_{k+p} = g_p(x_k)$ by $x_{l+1} = g_p(x_l)$ and obtain the piecewise linear system $x_{l+1} = f_{PWL}(x_l)$ with

$$x_{l+1} = \begin{cases} g_p(x_l), & \text{when } x_l \in D_p \\ 0, & \text{when } x_l \in D_\infty \cup [\mathcal{B} \setminus R_B] \end{cases} \tag{21}$$

where l can be seen as the new time variable, where we abstracted away from the time steps related to the motion inside $\mathcal{B}$. Some observations on the PWL system (21) are in order.

- We "completed" the piecewise linear model by adding dynamics to the system for the case when $x_l \in D_\infty$ and $x_l \in \mathcal{B} \setminus R_B$, so that it is defined completely on $\mathbb{R}^n$. In principle the dynamics on these sets are not important as will be proven below.
- A set D_p is in general not convex. It might even not be connected. See, the second example in section VIII.

Theorem 4.2: Consider system (7)-(9) *without* disturbances, F satisfies Assumption 4.1 and $\mathcal{B}$ is an open set containing the origin. Assume that there exists a $p_{max} < \infty$ such that (20) holds.

1) If the PWL system (21) is UB to the positively invariant Ω and $\mathcal{B} \subseteq \Omega$, then the *event-driven system* (7)-(9) is UB to Ω *on the event times*.
2) If the PWL system (21) has a convergence index $0 \; leq\lambda \leq 1$ to the positively invariant C-set Ω and $\mathcal{B} \subseteq \Omega$, then the *event-driven system* (7)-(9) has a convergence index λ to Ω *on the event times*.

Proof: Note that the system (7)-(9) on the event times is described by (12). We will use the latter system and the corresponding notation.

If $x_0 \in \Omega$ then we either have that the state trajectory of (12) satisfies $x_k \in\subseteq \Omega$ for all $k = 0, 1, 2, \ldots$ (which is in accordance with the properties of the theorem) or the state trajectory leaves Ω for some event time. Hence, we only have to consider the case where there exists a k_0 (take the smallest) for which $x_{k_0} \notin \Omega$ and thus $x_{k_0} \in \mathcal{B}^c = D_0$ because $\mathcal{B} \subseteq \Omega$. Note that the state x_k of system (12) never reaches $\mathcal{B} \setminus R_B$ for $k \geq k_0$ (by definition of R_B).

Observe that the dynamics of (12) and (21) coincide on $\bigcup_{i=0}^{\infty} D_i$ (modulo the motion inside $\mathcal{B}$, which lies in Ω by the hypothesis anyway). Hence, since $x_{k_0} \in D_0$, the system (12) follows the dynamics of (21) (modulo motion inside $\mathcal{B}$) for $k \geq k_0$ until D_∞ is reached - if ever (say at $k_1 \geq k_0$ with k_1 possibly equal to ∞). If D_∞ is reached, the state trajectory x_k of (12) stays inside $\mathcal{B} \subseteq \Omega$ for all $k \geq k_1$ by definition. Hence, on the time interval $[k_0, k_0+1, \ldots, k_1)$ the state of system (12) follows the motion of (21) and hence, the inheritance of the properties as described in the theorem is immediate. ■

Note that the larger p is, the more event times we are not updating the control value and thus we are not using the CPU for performing control computations. So, the larger p_{max} the more we can potentially save on computation time, but the complexer (the more regions) the resulting PWL model will be for the performance analysis. Fortunately, the computation of the ultimate bounds is performed off-line.

C. Including intersample behavior

The above results only provide statements *on the event times*. The behavior of the system in between the event times is not characterized. However, since at the event times we obtain properties like UB and convergence indices λ to a bounded set Ω we know that we enter Ω in finite time. Using this observation, an ultimate bound including the intersample behavior of (7) together with (8) or (9) can be computed from

$$x_{x_0,w}(t) - x_k = [e^{A_c(t-\tau_k)} - I]x_k + \\ + \int_{\tau_k}^{t} e^{A_c(t-\theta)} B_c u_k d\theta + \int_{\tau_k}^{t} e^{A_c(t-\theta)} E_c w(\theta) d\theta, \quad (22)$$

where $t \in [\tau_k, \tau_{k+1}]$.

In the non-uniform case we either have $x_{x_0,w}(t) \in \mathcal{B}$ or the equation (22) holds with $u_k = Fx_k$ and $\tau_{k+1} = \tau_k + T_s$. In the latter case using the boundedness of $\mathcal{W}_c$ we can easily see that $\|x_{x_0,w}(t) - x_k\| \leq CT_s(\|x_k\| + 1 + \|F\|\|x_k\|)$ for all $T_s \in [0, T_s^{max}]$. The constant $C = C(A_c, B_c, E_c, T_s^{max}, \mathcal{W}_c)$ depends on the system parameters, $\mathcal{W}_c$ and T_s^{max}. Hence, if the system (10) is UB to a set Ω, then the event-driven system (7)-(8) is UB to the set $\Omega \oplus B(0,\varepsilon)$ with $\varepsilon := \sup_{x\in\Omega} CT_s(\|x\| + 1 + \|F\|\|x\|)$ and $B(0,\varepsilon) := \{x \mid \|x\| \leq \varepsilon\}$. Note also that the convergence speed is maintained modulo the intersample behavior that can be bounded by a relative error around the trajectory on the event times.

In the uniform case the situation is a bit more complex. If $x_k \notin \mathcal{B}$ a similar bound as above can be derived. However, if $x_k \in \mathcal{B}$, then $u_k = Fx_r$ for some $r < k$, where $x_r = x_{x_0}(\tau_r) \notin \mathcal{B}$ and τ_r is the largest event time (smaller than τ_k) for which $x_{x_0,w}(\tau_r) \notin \mathcal{B}$. Hence, the quantity $\|x_{x_0,w}(t) - x_k\|$ is now bounded by $CT_s(\|x_k\| + 1 + \|F\|\|x_r\|)$. Note that x_r satisfies $(A+BF)x_r \in \mathcal{B}$. Since $\mathcal{B}$ is bounded and we have Assumption 4.1, this gives a bound on $\|x_r\|$. Consequently, we obtain a bound of $\|x_{x_0,w}(t) - x_k\|$ like $\tilde{C}T_s(\|x_k\| + \|F\| + 1)$ for some $\tilde{C}$. Note that if there are physical reasons that the control inputs are restricted to a bounded set, then we obtain immediately a bound like $CT_s(\|x_k\| + 1)$ independent of the designed controller gain F.

V. Computational aspects for the non-uniform case

There are several ways to compute RPI sets for discrete-time linear systems, see e.g. [4], [5], [8], [9], [12]. We will present here one approach based on ellipsoidal sets as in [9] to indicate how the derived results can be exploited.

To use the ellipsoidal approach of [9], we assume that $\mathcal{W}_d$ can be bounded by an ellipsoid $\mathcal{E}_{R^{-1}} := \{w \mid w^T R^{-1} w \leq 1\}$, $R > 0$. Techniques to find such an over-approximation are given in [6].

Along the lines of [9] it can be shown that feasibility of

$$P - \gamma^{-1} A_{cl} P A_{cl}^T - (1-\gamma)^{-1} R > 0 \quad (23)$$

for some $\gamma \in (0,1)$ yields (using Schur complements) that

$$(A_{cl}x + w)^T P^{-1} (A_{cl}x + w) < \gamma x^T P^{-1} x + (1-\gamma) w^T R^{-1} w.$$

From this it is easily seen that $x^T P^{-1} x \leq 1$ and $w^T R^{-1} w \leq 1$ imply $(A_{cl}x + w)^T P^{-1} (A_{cl}x + w) \leq 1$. This shows that $\Omega = \{x \mid x^T P^{-1} x \leq 1\}$ is a RPI set for (10). Moreover, we can show that the system (10) has a convergence factor $\lambda := \sqrt{\gamma + \frac{1-\gamma}{\mu}} < 1$ to $\sqrt{\mu}\Omega = \Omega(\mu) := \{x \mid x^T P^{-1} x \leq \mu\}$ with $\mu > 1$. This can be shown by taking $\mathcal{S} := \Omega$ and observing that the Minkowski functional $\Phi_{\mathcal{S}}(x)$ is equal to $\sqrt{x^T P^{-1} x}$. Hence, we have that

$$\begin{aligned} \Phi_{\mathcal{S}}^2(x_{k+1}) &= (A_{cl}x_k + w_k)^T P^{-1} (A_{cl}x_k + w_k) \\ &< \gamma x^T P^{-1} x + (1-\gamma) w^T R^{-1} w \\ &\leq \gamma \Phi_{\mathcal{S}}^2(x_k) + (1-\gamma). \end{aligned}$$

Since for $x_k \notin \text{int}\Omega(\mu)$ $\frac{\Phi_{\mathcal{S}}^2(x_k)}{\mu} \geq 1$, it follows that

$$\Phi_{\mathcal{S}}^2(x_{k+1}) \leq (\gamma + \frac{1-\gamma}{\mu}) \Phi_{\mathcal{S}}^2(x_k),$$

which shows that we have a convergence index $\lambda := \sqrt{\gamma + \frac{1-\gamma}{\mu}}$. Theorem 4.1 shows that we have a convergence index λ for the event-driven system (7)-(8) on the event times to $\Omega(\mu)$ if we select $\mu > 1$ and $\mathcal{B}$ such that $\text{cl}\mathcal{B} \subseteq \Omega(\mu)$.

VI. Computational aspects for the uniform case

Also for PWL systems several ways to compute invariant and contractive sets are available [10], [13]. We present here an approach based on ellipsoidal sets although techniques using reachability analysis can be exploited as well.

Theorem 6.1: Consider the event-driven system (7)-(9) without perturbations with F satisfying Assumption 4.1. Let $P > 0$ be a solution to $A_{cl}^T P A_{cl} - \gamma P < 0$ for some $\gamma \in (0,1)$. Take α^* small such that $\alpha^* > \max_{1,\ldots,p_{max}} \sup\{x^T P x \mid x \in g_p(D_p)\}$ and $\alpha^* > \max\{x^T P x \mid x \in \text{cl}\mathcal{B}\}$, where $g_p(D_p)$ denotes the image of the map g_p with its arguments in D_p. Define the set $\Omega(\alpha^*) := \{x \mid x^T P x \leq \alpha^*\}$. Then the PWL system (21) and consequently the event-driven system (7)-(9) on the event times have a convergence index $\sqrt{\gamma}$ to the set $\Omega(\alpha^*)$.

For brevity we omit the proof.

VII. Tuning of the controller

In this section we indicate how the ultimate bound Ω depends on $\mathcal{B}$ for (7), thereby facilitating the selection of desirable ultimate bounds by tuning $\mathcal{B}$. We will only present results for the non-uniform case due to space limitations. Similar results are available for the *unperturbed case* and uniform sampling. For the case of perturbed systems and uniform sampling, finding such relationships is still open.

The following result can be inferred from [4].

Theorem 7.1: Consider the system (7)-(8) with $\mathcal{W}_c$ a closed, convex set containing 0, F given and $\mathcal{B}$ an open set containing the origin.

- If Ω is a RPI set for the discrete-time linear system (10) containing $\text{cl}\mathcal{B}$, then for any $\mu \geq 1$ $\mu\Omega$ is a RPI set for (10) containing $\mu\text{cl}\mathcal{B}$.
- If the discrete-time linear system (10) is UB to Ω containing $\text{cl}\mathcal{B}$, then for any $\mu \geq 1$ (10) is UB to $\mu\Omega$ containing $\mu\text{cl}\mathcal{B}$
- If the discrete-time linear system (10) has convergence index $\lambda \leq 1$ to Ω containing $\text{cl}\mathcal{B}$, then for any $\mu \geq 1$ (10) has convergence index $\lambda \leq 1$ to $\mu\Omega$ containing $\mu\text{cl}\mathcal{B}$

This result shows that Ω scales "linearly" with $\mathcal{B}$ for scaling factors larger than one. Computing the minimal RPI set $\Omega_{\min}$ containing $\{0\}$, see e.g. [9] gives the ultimate bound as long as the chosen cl$\mathcal{B}$ lies inside $\Omega_{\min}$ the ultimate bound will stay constant and equal to $\Omega_{\min}$ (or strictly speaking an ultimate bound is the set $\mu\Omega_{\min}$ for any small $\mu > 1$). If cl$\mathcal{B}$ moves outside $\Omega_{\min}$, the linear scaling effect occurs. This effect is nicely demonstrated in the first example below.

VIII. Examples

A. Non-uniform sampling

To illustrate the theory in case of non-uniform sampling (8) we will use the example of the introduction. Note that in the introduction we used *uniform* sampling. In figure 4 the ratio of the number of control updates in comparison to the case where the updates are performed each sample time (i.e. $u_k = -0.45x_k$ for all x_k) and the maximal value of the state variable (after transients) $x_{\max} := \limsup_{t\to\infty} |x(t)|$ (the minimal ultimate bound), respectively, versus the parameter e_T are displayed, where $\mathcal{B} = \{x \mid |x| < e_T\}$.

The figure of the ultimate bounds can nicely be derived from the theory. First, we compute for the system (1), the discretized version (10) with sample time $T_s = 0.1$:

$$x_{k+1} = 1.051x_k + 1.025u_k + w_k;\ u_k = -0.45x_k \quad (24)$$

or

$$x_{k+1} = 0.590x_k + w_k \quad (25)$$

with $3.076 \leq w_k \leq 3.076$. The minimal RPI set $\Omega_{\min}$ for (25) containing $\{0\}$ is equal to the ellipsoid $[-7.50, 7.50]$. Hence, note that as long as $e_T < 7.50$ the ultimate bound of the system (7)-(8) is equal to $\Omega_{\min}$ (or strictly speaking to the set $\mu\Omega_{\min}$ for a small $\mu > 1$ as discussed in Section V). This explains the constant line in the $x_{\max}$ versus e_T plot in Figure 4 up to $e_T = 7.50$. At the moment e_T gets greater than 7.50, the condition of theorem 4.1 that cl$\mathcal{B} \subset \Omega_{\min}$ does no longer hold. However, we can now use the "scaling effect" from Theorem 7.1. Theorem 7.1 implies that $\frac{e_T}{7.50}\Omega_{\min}$ is RPI and the linear system is UB to $\mu\frac{e_T}{7.50}\Omega_{\min}$ for any $\mu > 1$ when $e_T > 7.50$. Since cl$\mathcal{B} \subseteq \frac{e_T}{7.50}\Omega_{\min}$ holds, Theorem 4.1 implies that $\frac{e_T}{7.50}\Omega_{\min}$ is RPI and the event-driven system (7)-(8) is UB to $\mu\frac{e_T}{7.50}\Omega_{\min}$ for any $\mu > 1$. This explains the linear part in the $x_{\max}$ versus e_T plot in Figure 4. Hence, we can reduce the number of control updates with almost 80% in this set-up without reducing the control accuracy (e.g. take $e_T = 5$)!

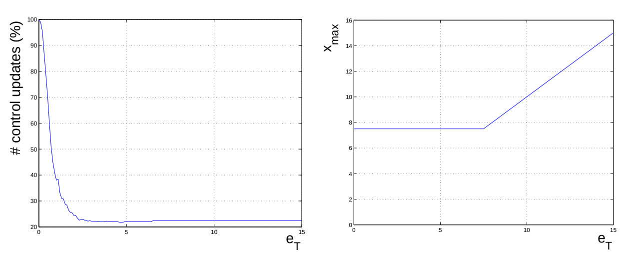

Fig. 4. e_T versus the control effort and $x_{\max}$ for system (1)-(2).

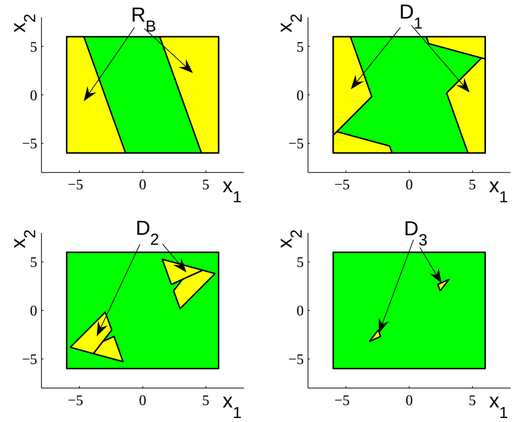

Fig. 5. Sets R_B, D_1, D_2, D_3 in yellow (light grey), set $\mathcal{B}$ the rectangle underneath the sets D_p in green (dark grey).

B. Uniform sampling

To demonstrate the results for uniform sampling, we have taken the example of an unstable system with two states ($n = 2$) given by (7) with

$$A_c = \begin{bmatrix} 1070 & 270 \\ 270 & 40 \end{bmatrix};\ B_c = \begin{bmatrix} 453 \\ 874 \end{bmatrix} \quad (26)$$

The controller matrix is taken to be $F = [-2.4604\ -0.2340]$. The matrices in the discrete-time version (10) are equal to

$$A = \begin{bmatrix} 3.00 & 0.50 \\ 0.50 & 1.10 \end{bmatrix} \quad B = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix} \quad (27)$$

for $T_s = 0.001$. Note that the the eigenvalues of $A_{cl} = A + BF$ are 0.7 ± 0.7. $\mathcal{B} = \{x \mid |x_1| < e_T, |x_1| < e_T\}$ with $e_T = 6$. One can easily check that the conditions mentioned in Remark 4.1 are satisfied. Hence, $D_\infty = \emptyset$ and the finite $p_{\max}$ that one finds is equal to 3. Figure 5 displays the calculated sets R_B and D_p, $p = 1, 2, 3$ as given by equation (18)

The dynamics that are valid inside D_p, calculated with equation (17) are:

$$\begin{aligned} g_0(x_l) &= \begin{bmatrix} 0.537 & 0.264 \\ -1.96 & 0.863 \end{bmatrix} x_l \\ g_1(x_l) &= \begin{bmatrix} 0.364 & 1.03 \\ -2.13 & 1.63 \end{bmatrix} x_l \\ g_2(x_l) &= \begin{bmatrix} -2.60 & 4.43 \\ -4.79 & 2.83 \end{bmatrix} x_l \\ g_3(x_l) &= \begin{bmatrix} -12.8 & 15.2 \\ -9.17 & 5.84 \end{bmatrix} x_l \end{aligned} \quad (28)$$

Since we have obtained the PWL-description of the system we can apply the theory presented in section VI. Using the ellipsoidal approach as presented in Theorem 6.1 we obtain the ellipsoid Ω in figure 6. We also computed the reachable set Ω_{reach} for the PWL system from points in R_B. For the computation of this set a combination of tools from [10] and [8] was used. Note that Ω_{reach} is a positive invariant set for the PWL system. Since $\mathcal{B} \subset \Omega_{\text{reach}}$ and outside $\mathcal{B}$

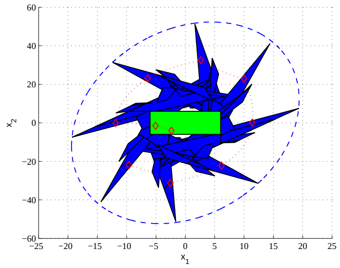

Fig. 6. Ellipsoid Ω indicated by the dashed (blue) line, set Ω_{min} in blue (dark grey) and set $\mathcal{B}$ in green (light grey).

the dynamics on the event times is equal to $x_{k+1} = A_{cl}x_k$, similar statements can be made for Ω_{reach} as for Ω.

Figure 6 also shows a time simulation of the continuous time system. A (red) dotted line shows the intersample behavior in which the small (red) diamonds indicate the values at the event times. It can be seen that the trajectory is not restricted to the depicted Ω_{reach} (in blue (dark grey)), due to the intersample behavior. Bounds on the intersample behavior can be obtained via Section IV-C.

IX. Conclusions

This paper advocates the use of event-driven controllers to reduce the required (average) processor load for the implementation of digital controllers. An initial example already illustrated the reduction of control computations (up to 80%) that is achievable. In [15] it is experimentally studied how this reduction in control computations is related to lowering the average processor load. However, the trade-off one has to make is to balance this reduction with the control accuracy. This paper provides necessary theory to get insight in this trade-off and shows the ultimate bounds that are obtainable and how they depend on the parameters of the control strategy. The theory is based on inferring properties (like robust positive invariance, ultimate boundedness and convergence indices) for the event-driven controlled system from discrete-time linear systems (in case of non-uniform sampling) or piecewise linear systems (in case of uniform sampling). Although this paper analyses a rather simple event-driven control structure, it already indicates the complexity and challenge for the analysis and synthesis of these type of control loops. This work provides the first step in a proper analysis of these types of loops and future work will be focussed on the finite number of regions of the piecewise linear model (finite p_{max}) and on extensions (e.g. reference tracking).

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